Integration by Substitution

Remember your favorite topic from first semester? That's right, the chain rule. ☺ Let's review by doing some examples:

Example: $f(x) = (x^3 + 1)^5$

$$f'(x) = 5(x^3 + 1)^4 (3x^2)$$

Example: $f(x) = \sin 5x$

$$f'(x) = \cos 5x \cdot (5)$$

Example: $f(x) = (x^2 + 3)^7$

$$f'(x) = 7(x^2 + 3)^6 (2x)$$

Since the chain rule involves a pattern, we should be able to do it in reverse.

Example: Find $\int 2x(x^2+4)^2\,dx$.

Notice that $2x$ is the derivative of $x^2 + 4$. We know the integral of $(\ \ )^2$ is $\frac{1}{3}(\ \ )^3$. So,

$$\int 2x(x^2+4)^2\,dx = \frac{1}{3}(x^2+4)^3 + C.$$

Let's check our answer by taking the derivative:

$$\frac{d}{dx}\left[\frac{1}{3}(x^2+4)^3\right] = (x^2+4)^2(2x)$$

What we have to notice is the derivative of the inner piece.

The inner pieces we look for can be many things, but for starters, we will look for the following:

1. Stuff inside parentheses to a power
2. Stuff under the radical
3. Stuff in the arguments of trig functions

Formally, it looks like this:

Chain Rule: $\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x)$

This also works with the antiderivative, F.

$$\frac{d}{dx} F(g(x)) = F'(g(x))g'(x) = f(g(x))g'(x)$$

So,

$$\int f(g(x))g'(x) = F(g(x)) + C.$$

Let's go back to our original example:

$$\int \underbrace{2x}_{g'(x)}\overbrace{(\underbrace{x^2+4}_{g(x)})^2}^{f(g(x))}dx$$

Sometimes, this can be difficult to see, so let's look at the formal process: u substitution. We will let $u = g(x)$.

$$\int 2x(x^2+4)^2\,dx$$

Let $u = x^2 + 4$.

Then $du = 2x\,dx$.

Now, we can rewrite the integral in terms of u, instead of in terms of x, and integrate. Then we put the answer back in terms of x.

$$\begin{aligned}\int u^2 du &= \frac{1}{3}u^3 + C \\ &= \frac{1}{3}(x^2+4)^3 + C\end{aligned}$$

Let's do some examples:

Example: $\int 5\cos 5x\,dx$

Let $u = 5x$.

$du = 5\,dx$

$$\int \cos u\,du = \sin u + C$$

$$= \sin 5x + C$$

Example: $\int \cos 5x\,dx = \frac{1}{5}\int 5\cos 5x\,dx$

Let $u = 5x$.

$du = 5\,dx$

$$\frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C$$

$$= \frac{1}{5}\sin 5x + C$$

There is no 5 in the original, but we can put in the constant, as long as we balance it out by multiplying by one-fifth.

Example: $\int x^2\sqrt{x^3+1}\,dx$

Let $u = x^3 + 1$.

$du = 3x^2 dx$

$$\int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(x^3+1)^{3/2} + C$$

Example: $\int \frac{x^3}{\sqrt{1+x^4}}\,dx = \frac{1}{4}\int \frac{4x^3}{\sqrt{1+x^4}}\,dx$ Let $u = 1 + x^4$.

$du = 4x^3 dx$

$$\frac{1}{4}\int \frac{1}{\sqrt{u}}\,du = \frac{1}{4}\int u^{-1/2}du$$

$$= \frac{1}{2}u^{1/2} + C$$

$$= \frac{1}{2}(1+x^4)^{1/2} + C$$

Example: $\int \tan(3x)\sec^2(3x)\,dx$ Let $u = \tan(3x)$.

$du = 3\sec^2(3x)dx$

On this one, notice that the derivative of tangent is secant squared.

$$\frac{1}{3}\int 3\tan(3x)\sec^2(3x)\,dx = \frac{1}{3}\int u\,du$$

$$= \frac{1}{3}\cdot\frac{1}{2}u^2 + C$$

$$= \frac{1}{6}u^2 + C$$

$$= \frac{1}{6}\tan^2(3x) + C$$

Example: $\int x\sqrt{2x-1}\,dx$

Let $u = 2x-1$.

$du = 2\,dx$

When we go through the process of u-substitution in this problem, we have to deal with an extra x. We can write it in terms of u, by solving our u equation for x.

$$u = 2x-1$$

$$u+1 = 2x$$

$$x = \frac{u+1}{2}$$

$$\int\left(\frac{u+1}{2}\right)\sqrt{u}\,du = \frac{1}{2}\int(u+1)\sqrt{u}\,du$$

$$= \frac{1}{2}\int(u^{3/2}+u^{1/2})du$$

$$= \frac{1}{2}\left(\frac{2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+C$$

$$= \frac{1}{5}(2x-1)^{5/2}+\frac{1}{3}(2x-1)^{3/2}+C$$

Sometimes, when we are doing more difficult stuff, we tend to overthink some things. Be careful not to make the problem any more difficult than it really is.

Example: $\int (x^2+1)^2 dx$

There's no need to do u-substitution; we just expand the binomial.

$$\int (x^4+2x^2+1)dx$$

$$\frac{1}{5}x^5+\frac{1}{3}x^3+x+C$$