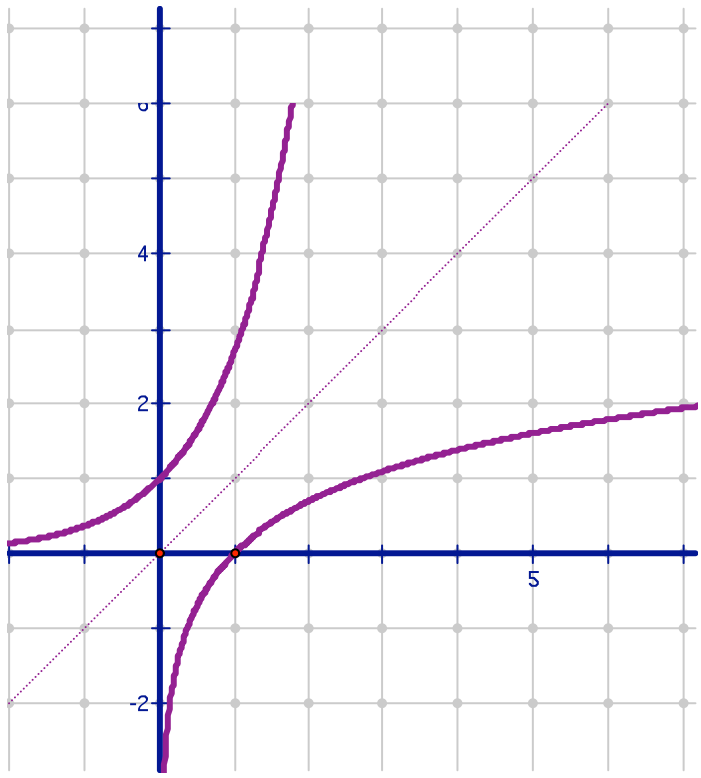
Differentiating and Integrating Exponential Functions

Inverse of the Natural Logarithm



If $f(x) = \ln x$ and $f(f^{-1}(x)) = x$, then $\ln (f^{-1}(x)) = x$.

Consider $\ln e^x = x \ln e = x$. Therefore, $f^{-1}(x) = e^x$.

Definition: e^x is the inverse of $\ln x$. In other words,

$$y = e^x \text{ iff } x = \ln y$$

Basic Rules

$$\ln e^x = x$$

$$e^{\ln x} = x$$

Example 1: $e^{x+1} = 7$

$$\ln e^{x+1} = \ln 7$$
$$(x+1)\ln e = \ln 7$$
$$x+1 = \ln 7$$
$$x = \ln 7 - 1$$

Example 2: $8 = 10^{x+5}$

$$\log 8 = \log 10^{x+5}$$
$$\log 8 = (x+5)\log 10$$
$$\log 8 = x+5$$
$$\log 8 - 5 = x$$

Example 3: $\ln(2x-3) = 5$

$$e^{\ln(2x-3)} = e^5$$
$$2x-3 = e^5$$
$$2x = e^5 + 3$$
$$x = \frac{e^5+3}{2}$$

Now let's find the derivative of $y = e^x$.

$$y = e^x$$

$$\ln y = \ln e^x$$

Take the natural log of both sides.

$$\ln y = x$$

$$\frac{1}{y} y' = 1$$

Differentiate with respect to x.

$$y' = y$$

$$y' = e^x$$

Pretty cool! The derivative of e^x is e^x. Of course, we also have the Chain Rule version.

$$y = e^u$$

$$y' = e^u \cdot u'$$

Example 4: $\frac{d}{dx}e^{5x^2}$

$$e^{5x^2}\cdot 10x$$

$$10xe^{5x^2}$$

Example 5: $\frac{d}{dx}e^{\sin x}$

$$e^{\sin x}\cdot\cos x$$

$$\cos x\, e^{\sin x}$$

Example 6: $\frac{d}{dx}e^{\ln x}$

$$e^{\ln x}\cdot\frac{1}{x}$$

$$x\cdot\frac{1}{x}$$

$$1$$

What about integrating exponential functions?

Well, if the derivative of e^x is e^x, then so is the integral.

$$\int e^x dx = e^x + C$$

Example 7: $\int 3x^2 e^{x^3} dx$

Let $u = x^3$
$du = 3x^2\, dx$

$$\int 3x^2 e^{x^3} dx = \int e^u du$$
$$= e^u + C$$
$$= e^{x^3} + C$$

Example 8: $\int_{\pi/6}^{\pi/4} \sec^2 x e^{\tan x} dx$

Let $u = \tan x$
$du = \sec^2 x\, dx$

$$x = \frac{\pi}{4} \Rightarrow u = \tan\frac{\pi}{4} = 1$$
$$x = \frac{\pi}{6} \Rightarrow u = \tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$$

$$\int_{\pi/6}^{\pi/4} \sec^2 x e^{\tan x} dx = \int_{\sqrt{3}/3}^{1} e^u du$$
$$= e^u \Big]_{\sqrt{3}/3}^{1}$$
$$= e^1 - e^{\sqrt{3}/3}$$