Bases Other Than e

Today we want to go from e^x to a^x.

Consider $x = e^{\ln x}$. So $a^x = e^{\ln a^x}$.

$$\boxed{a^x = e^{x \ln a}}$$

This is the calculus definition.

Also, $y = \ln x \Leftrightarrow e^y = x$.

So, $y = \log_a x \Leftrightarrow a^y = x$.

By switching x and y, we get $y = a^x$.

So, $\log_a x$ is the inverse of a^x.

Therefore,

$$\log_a a^x = x \qquad \text{and} \qquad a^{\log_a x} = x.$$

Now, consider the following:

$$y = \log_a x.$$

This means

$$a^y = x.$$

Take the natural log of both sides.

$$\ln a^y = \ln x$$

Use the Exponent Rule.

$$y \ln a = \ln x$$

$$y = \frac{\ln x}{\ln a}$$

You could also use the Change of Base Formula.

What about the calculus? How do we find $\frac{d}{dx}a^x$?

$$y = a^x$$

$$\ln y = \ln a^x$$

$$\ln y = x \ln a$$

$$\frac{1}{y} \cdot y' = \ln a$$

$$y' = y \ln a$$

$$y' = a^x \ln a$$

Chain Rule version:

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}$$

How about $\frac{d}{dx}\log_a x$?

$$\begin{aligned}\frac{d}{dx}\left(\frac{\ln x}{\ln a}\right) &= \frac{d}{dx}\left(\frac{1}{\ln a}\cdot\ln x\right)\\ &= \frac{1}{\ln a}\cdot\frac{1}{x}\\ &= \frac{1}{x\ln a}\end{aligned}$$

Chain Rule version:

$$\frac{d}{dx}\log_a u = \frac{1}{u\ln a}\frac{du}{dx}$$

Find the derivative of each of the following.

Example 1: $\frac{d}{dx}10^x$

$$\frac{d}{dx}10^x = 10^x \ln 10$$

Example 2: $y = 3^{x^2}$

$$y' = 3^{x^2}(\ln 3)\cdot(2x)$$

Example 3: $\frac{d}{dx}\log_3 x$

$$\frac{d}{dx}\log_3 x = \frac{1}{x\ln 3}$$

Example 4: $y = \log_{10} 8x^2$

$$y' = \frac{1}{8x^2 \ln 10}\cdot 16x$$

$$= \frac{2}{x\ln 10}$$

How about integrating these functions?

$$\int a^x dx = \frac{1}{\ln a} \cdot a^x + C$$

$$\int a^u du = \frac{1}{\ln a} \cdot a^u + C$$

Find each of the following.

Example 5: $\int 3x^2 \cdot 5^{x^3}\, dx$

$$\int 3x^2 \cdot 5^{x^3}\, dx = \int 5^u\, du$$

Let $u = x^3$
$du = 3x^2\, dx$

$$= \frac{1}{\ln 5} \cdot 5^u + C$$

$$= \frac{1}{\ln 5} \cdot 5^{x^3} + C$$

Example 6: $\int \cos x \cdot 4^{\sin x}\, dx$

$$\int \cos x \cdot 4^{\sin x}\, dx = \int 4^u\, du$$

Let $u = \sin x$
$du = \cos x\, dx$

$$= \frac{1}{\ln 4} \cdot 4^u + C$$

$$= \frac{1}{\ln 4} \cdot 4^{\sin x} + C$$