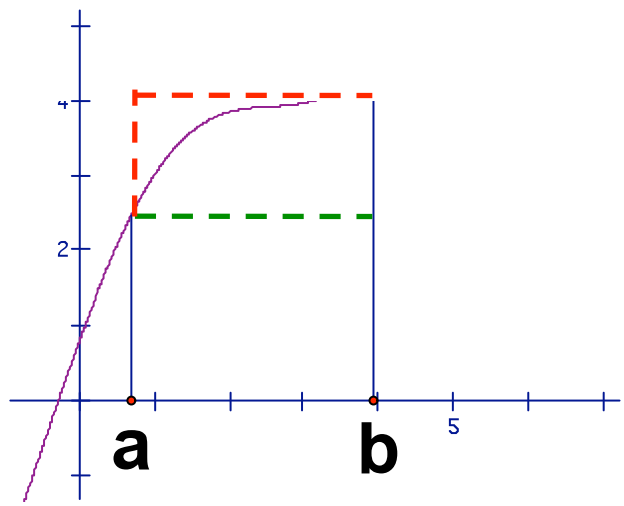
Mean Value Theorem

Consider the following curve and the area bounded by the x-axis, y-axis, $x = a$ and $x = b$.



Area of the small rectangle: $f(a)(b - a)$

height width

Area of the large rectangle: $f(b)(b - a)$

Somewhere in between, there's a value of c such that $f(c)$ is the height of the rectangle whose area is equal to the area under the curve.

$$\int_a^b f(x)dx = f(c)(b-a)$$

This is the **Mean Value Theorem (MVT)** for integrals (p. 278):

If f is continuous on $[a, b]$, then there exists a c in $[a, b]$ such that $\int_a^b f(x)dx = f(c)(b-a)$.

The value $f(c)$ is also called the average value of the function on $[a, b]$.

$$f(c) = \frac{1}{b-a}\int_a^b f(x)dx$$

We can use this to find the average velocity, average profit, etc., if we just integrate the function and divide by $b - a$.

Example: Find the average value of $f(x) = x^3 + 1$ on $[0, 2]$ and find the x value (c) where it occurs.

$$\begin{aligned}\frac{1}{2-0}\int_0^2 (x^3+1)\,dx &= \frac{1}{2}\left[\frac{1}{4}x^4 + x\right]_0^2 \\ &= \frac{1}{2}[(4+2)-0] \\ &= \frac{1}{2}(6) \\ &= 3\end{aligned}$$

So, the average value, $f(c)$, is 3. We solve the following equation to get the value of c.

$$\begin{aligned}f(c) &= c^3 + 1 \\ 3 &= c^3 + 1 \\ c^3 &= 2 \\ c &= \sqrt[3]{2}\end{aligned}$$