Riemann Sums

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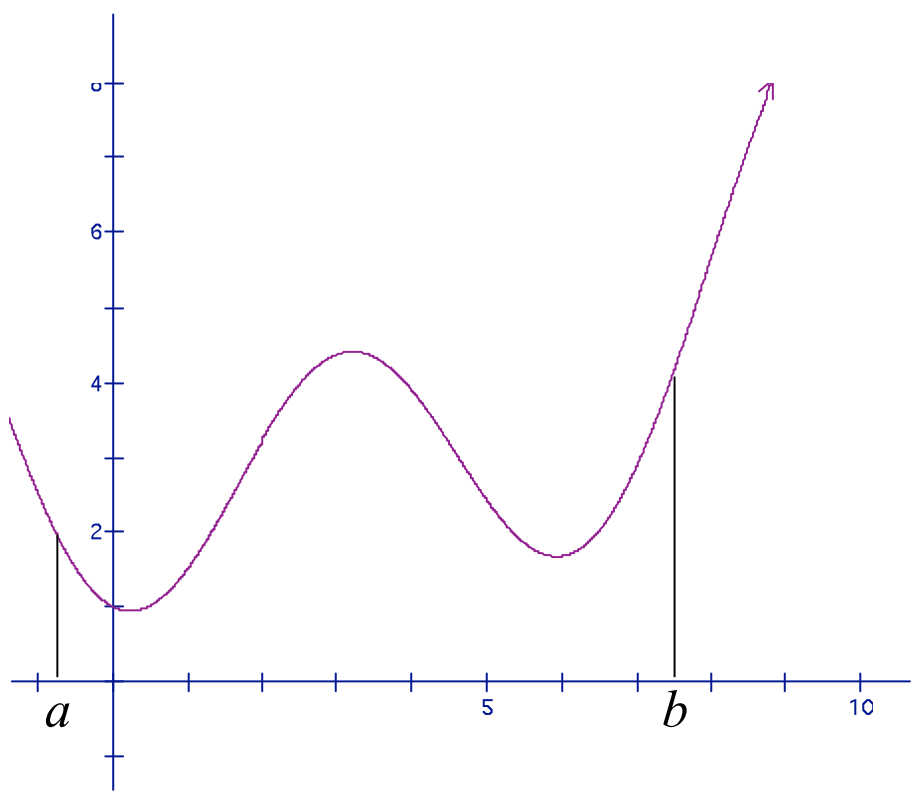
Yesterday, all my troubles seemed so far away, and we decided Δx needed to be constant. Today, thanks to GFBR, we can let Δx vary.

Consider the function below in which the interval $[a, b]$ is partitioned into n subintervals where

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$



Now let's deal with the height of the rectangle. Previously we used the minimum or the maximum each time. Today, we can use any height in the subinterval: minimum, maximum, middle, random, anywhere.

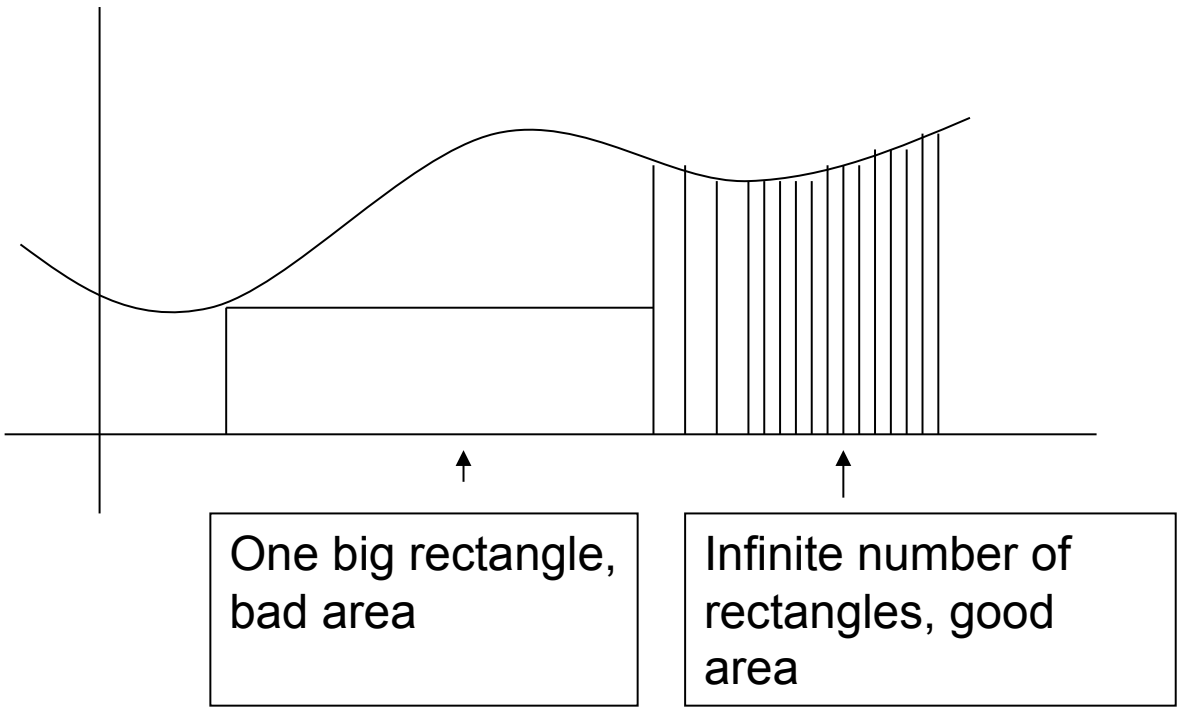
Riemann Sum: Let f be defined on $[a, b]$ and let Δx be a partition of $[a, b]$ given by $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$ where Δx_i is the width of the ith subinterval and c_i is any point in the ith subinterval. The following sum is called the Riemann Sum of f.

$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$

To increase the precision, we increase the number of rectangles. An infinite number of rectangles would be nice. However,

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

is not good enough, since all partitions are not the same; one big fat rectangle could stay big with an infinite number of skinny ones (see graph on the next page). To take care of this, we use the norm!



The norm (notation on the following page) refers to the width of the largest rectangle. So instead of taking n to infinity, we will shrink the norm to zero.

The **norm**, $||\Delta||$, of a partition Δ is the length of the longest subinterval.

Instead of letting $n \to \infty$, let $||\Delta|| \to 0$. This will give us the exact area.

$$\lim_{\|\Delta\| \to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx$$

This is the definition of the definite integral. You need to know this.

Also, this integral

$$\int_a^b f(x)dx$$

is the area of the region bounded by f, the x-axis, $x = a$ and $x = b$ if f is continuous and nonnegative.

How do we do calculate definite integrals?

Drum roll, please……

☺ The Fundamental Theorem of Calculus ☺

If f is continuous on $[a, b]$ and F is any antiderivative (so we choose $c = 0$) of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a).$$

It can also be written as:

$$\int_a^b f(x)dx = F(x)\Big|_a^b$$

Let's look again at an example of finding the sum of a region from the previous section:

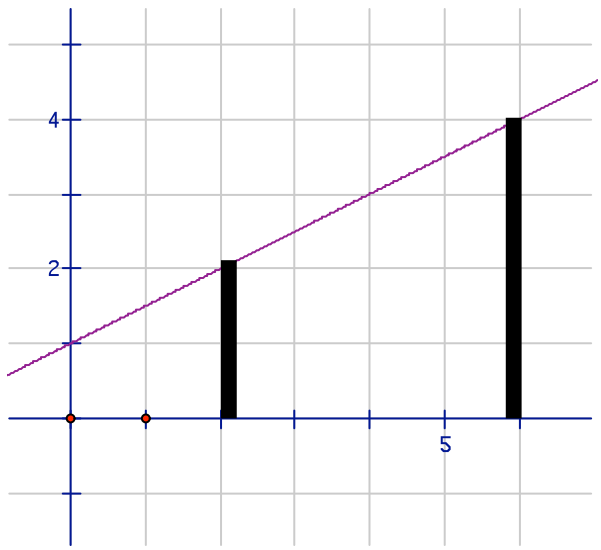
$$f(x) = x^2 + 1 \text{ on } [0, 2]$$

In that section, we discovered that the lower sum was 3.75, and the upper sum was 5.75. Now let's find the sum using the definite integral.

$$\begin{aligned}
\int_0^2 (x^2+1)\,dx &= \left(\frac{1}{3}x^3 + x\right)\Bigg|_0^2 \\
&= \left[\frac{1}{3}(2)^3 + 2\right] - \left[\frac{1}{3}(0)^3 + 0\right] \\
&= \frac{8}{3} + 2 \\
&= \frac{14}{3} = 4\frac{2}{3}
\end{aligned}$$

exact area

Example 1: Find the area on $[2, 6]$ for the function $f(x) = \frac{1}{2}x + 1$.



From algebra, we know that the area is 12:

a 2-by-4 rectangle and a 4-by-2 triangle.

Let's see if calculus gets us the same answer.

$$\int_2^6 \left(\frac{1}{2}x + 1\right) dx = \left(\frac{1}{4}x^2 + x\right)\Big|_2^6$$

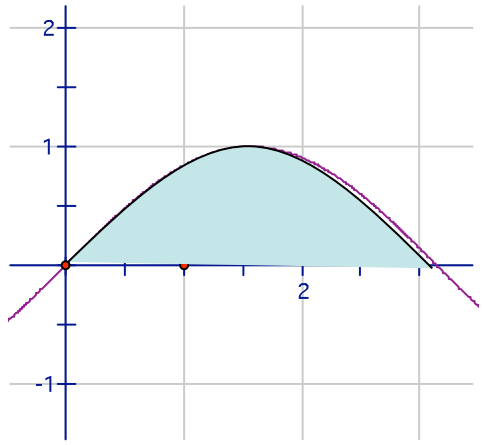
$$= \left(\frac{1}{4} \cdot 6^2 + 6\right) - \left(\frac{1}{4} \cdot 2^2 + 2\right)$$

$$= 15 - 3$$

$$= 12$$

Yes, we get the same thing.

Example 2: For the function $y = \sin x$, find the area on $[0, \pi]$.



$$\int_0^{\pi} \sin x \, dx = -\cos x \Big|_0^{\pi}$$

$$= -\cos x - (-\cos 0)$$

$$= 1 + 1$$

$$= 2$$

Now that is a pretty cool result.