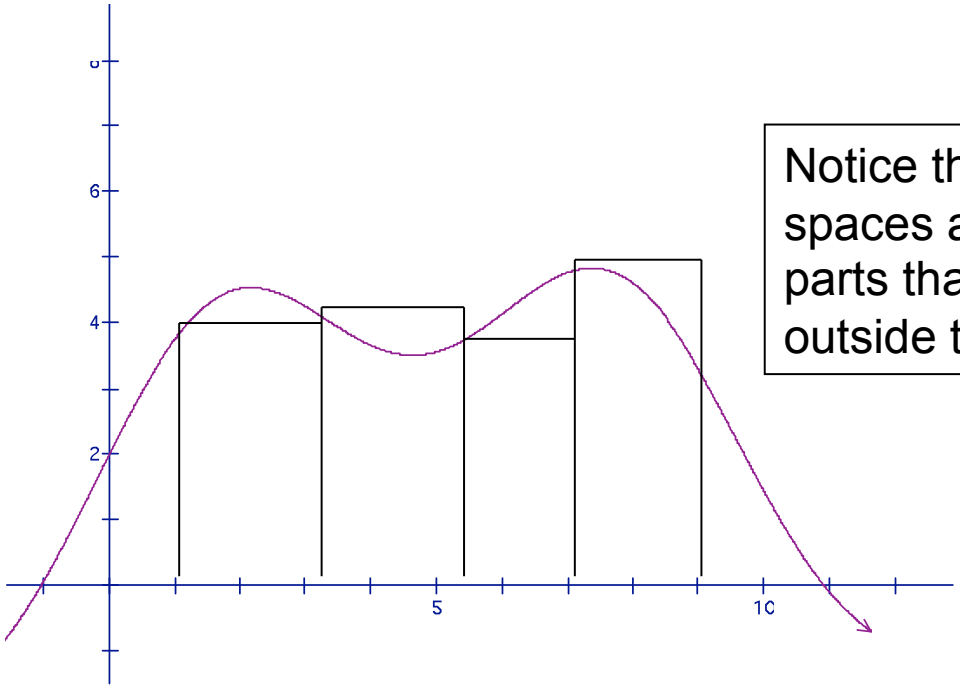
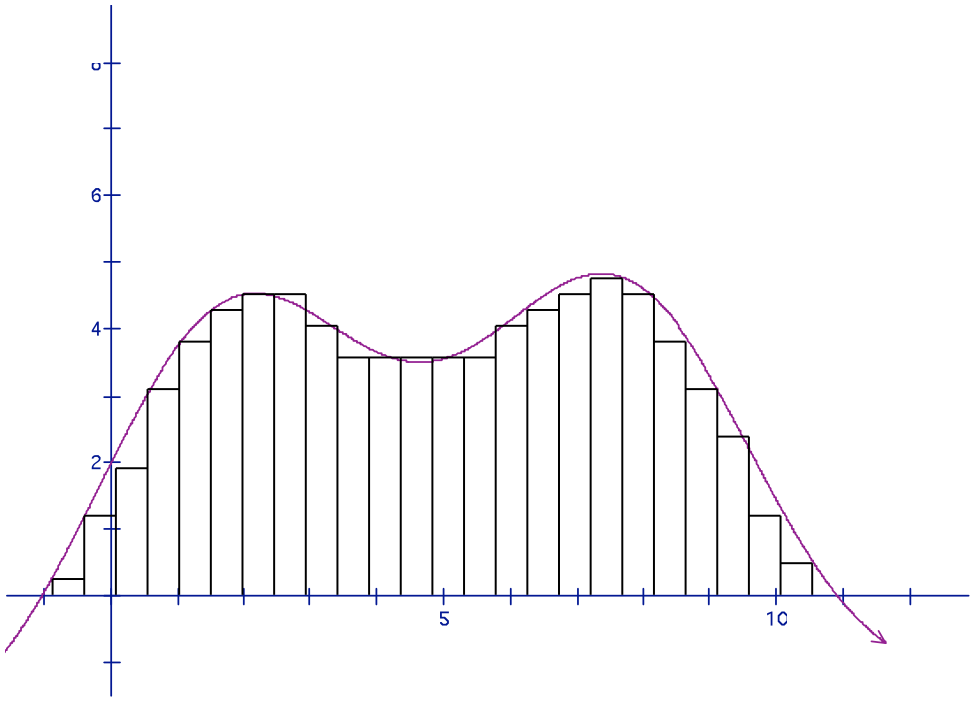
Sigma Notation and Area Under a Curve

Let's consider the area under the following curve.



Notice the missing spaces and the parts that go outside the area.

The widths of these rectangles don't give us a very good approximation of the area.



The more rectangles we have, the more accurate we can be. An infinite number of rectangles would be nice, but we will get to that later.

Since we are going to be adding things up, let's review sigma notation.

$$\sum_{i=1}^{5} a_i = a_1 + a_2 + a_3 + a_4 + a_5$$

$$\sum_{j=1}^{5} j = 1 + 2 + 3 + 4 + 5 = 15$$

$$\sum_{i=1}^{n} 3i = 3(1) + 3(2) + 3(3) + \cdots + 3(n)$$

$$\sum_{i=1}^{100} \frac{1}{i^2} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{99^2} + \frac{1}{100^2}$$

$$\sum_{i=1}^{n} f(x_i)\Delta x_i = f(x_1)\Delta x_1 + f(x_2)\Delta x_2 + \cdots f(x_n)\Delta x_n$$

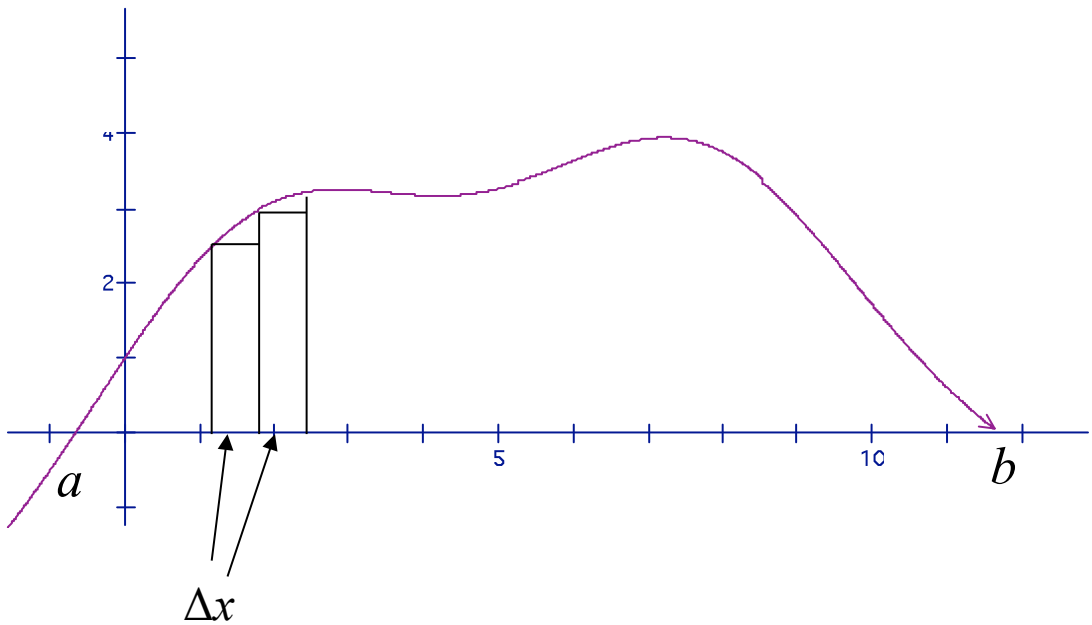
This last one looks difficult to calculate, but if all the Δx's are the same, we can factor it out and it makes it simpler.

Properties of Sigma Notation

$$\sum_{i=1}^{n} ka_i = k\sum_{i=1}^{n} a_i \qquad \sum_{i=1}^{n}(a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$$

What do we need to know to find the area of a rectangle?

We need to know the base and the height.



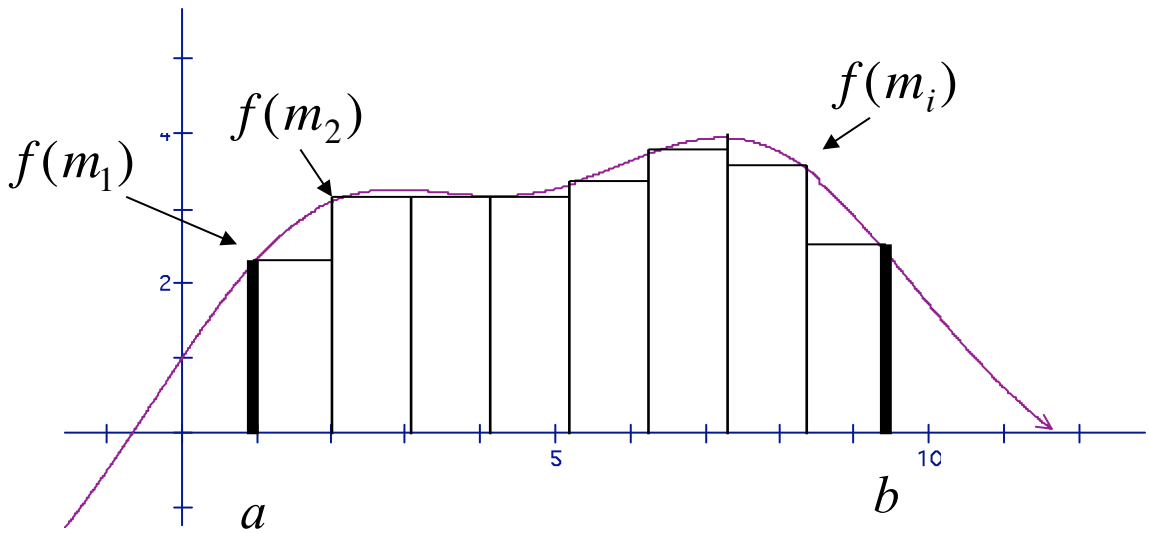
To keep it simple, we will use n equal partitions, each Δx. To create these partitions, we use

$$\Delta x = \frac{b-a}{n}.$$

So the base of each rectangle we will use will be Δx. This will make doing the sum easier, as mentioned on the previous page.

This takes care of the base, what about the height?

We will start off by considering the lower sum, using the minimum value on the subinterval $[a, b]$.



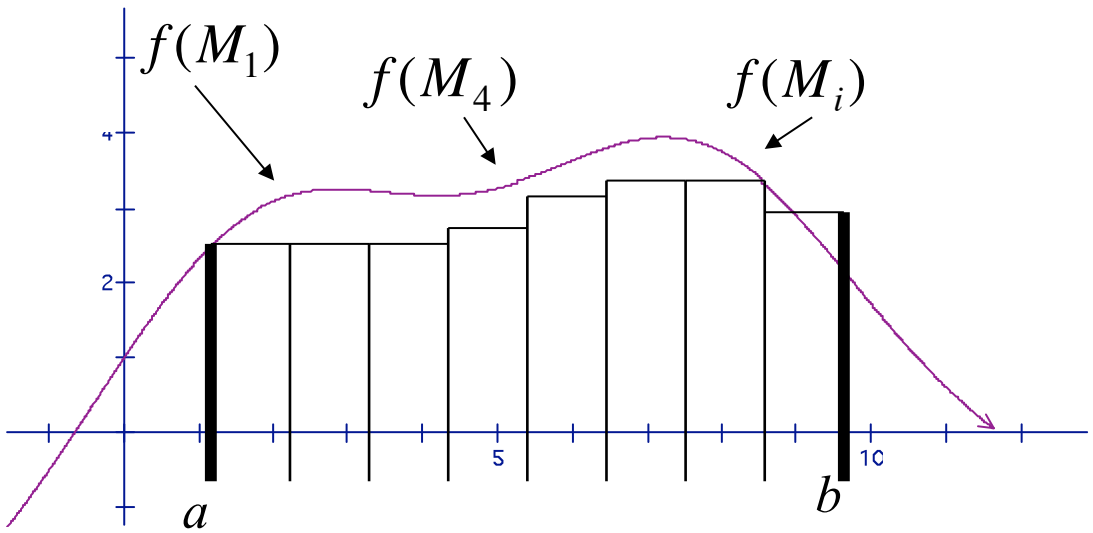
From the graph above, we can see that the area will be

$$L = f(m_1)\Delta x + f(m_2)\Delta x + f(m_3)\Delta x + \cdots + f(m_n)\Delta x.$$

Or, in sigma notation,

$$L = \sum_{i=1}^{n} f(m_i)\Delta x.$$

Now let's consider the upper sum, where the height will be the maximum value on the subinterval.



From this graph, we can see that the area will be

$$U = f(M_1)\Delta x + f(M_2)\Delta x + f(M_3)\Delta x + \cdots + f(M_n)\Delta x.$$

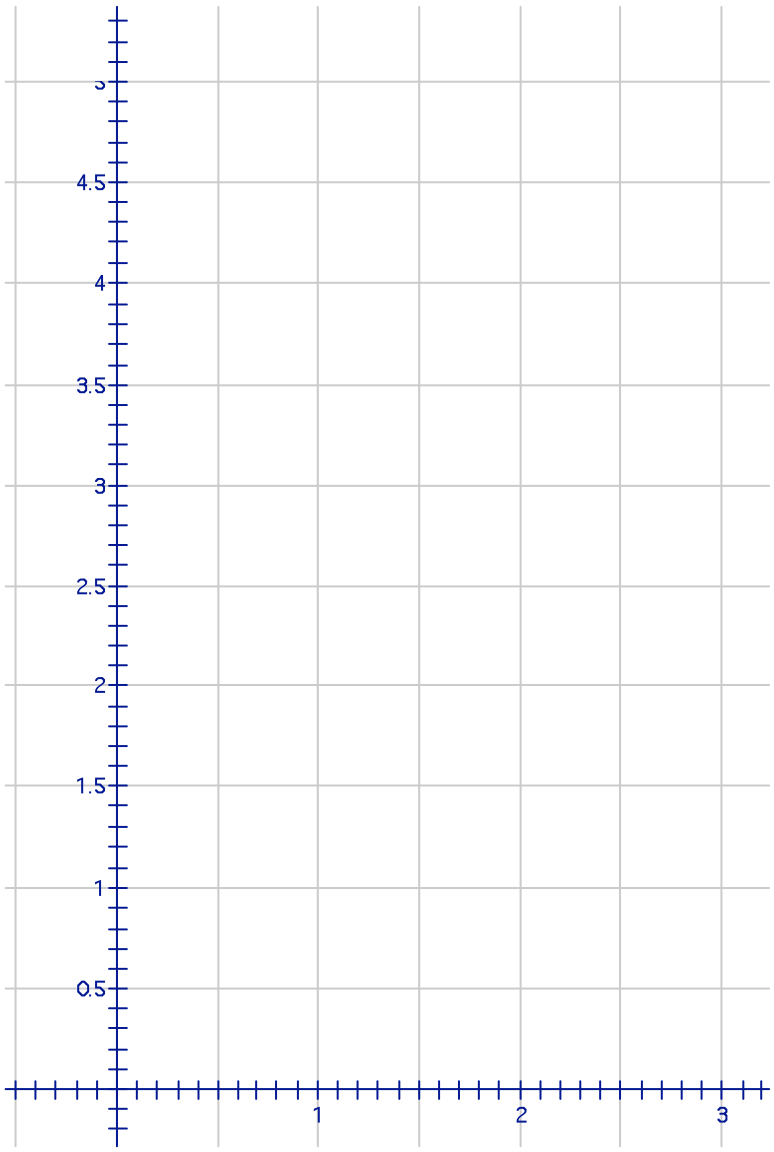
Or, in sigma notation,

$$U = \sum_{i=1}^{n} f(M_i)\Delta x.$$

How do the lower and upper sums compare to the actual area?

$$L < \text{actual area} < U$$

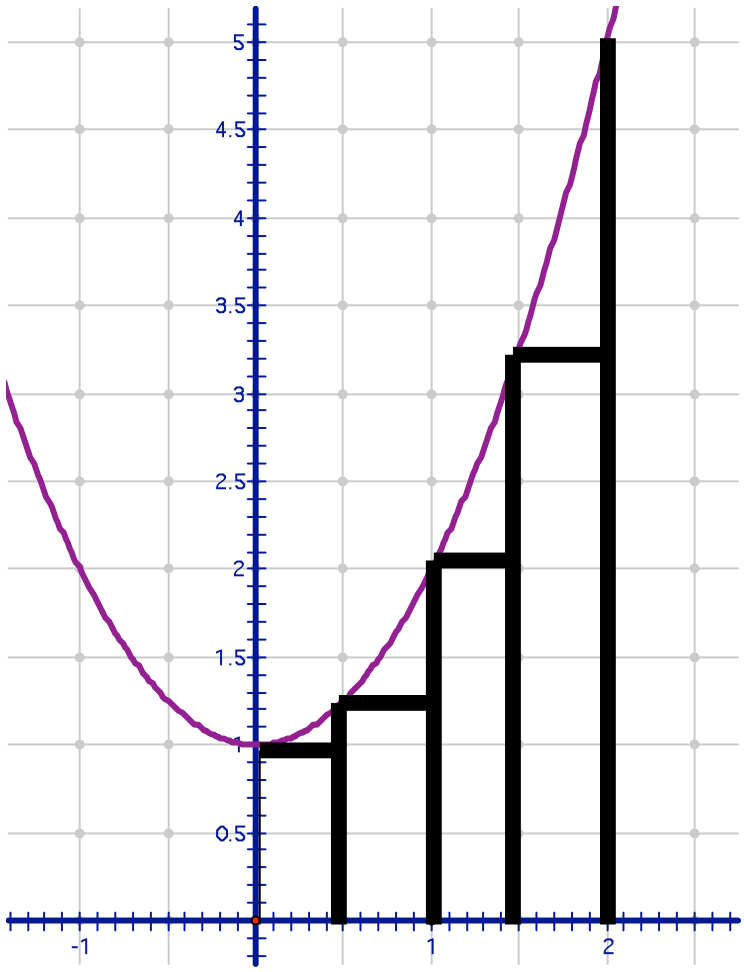
Example: Find the lower and upper sum of the region bounded by $f(x) = x^2 + 1$ and the x-axis. Let $n = 4$ on $[0, 2]$.



$$\Delta x = \frac{2-0}{4} = \frac{1}{2}$$

This will be the width of each rectangle.

Lower Sum



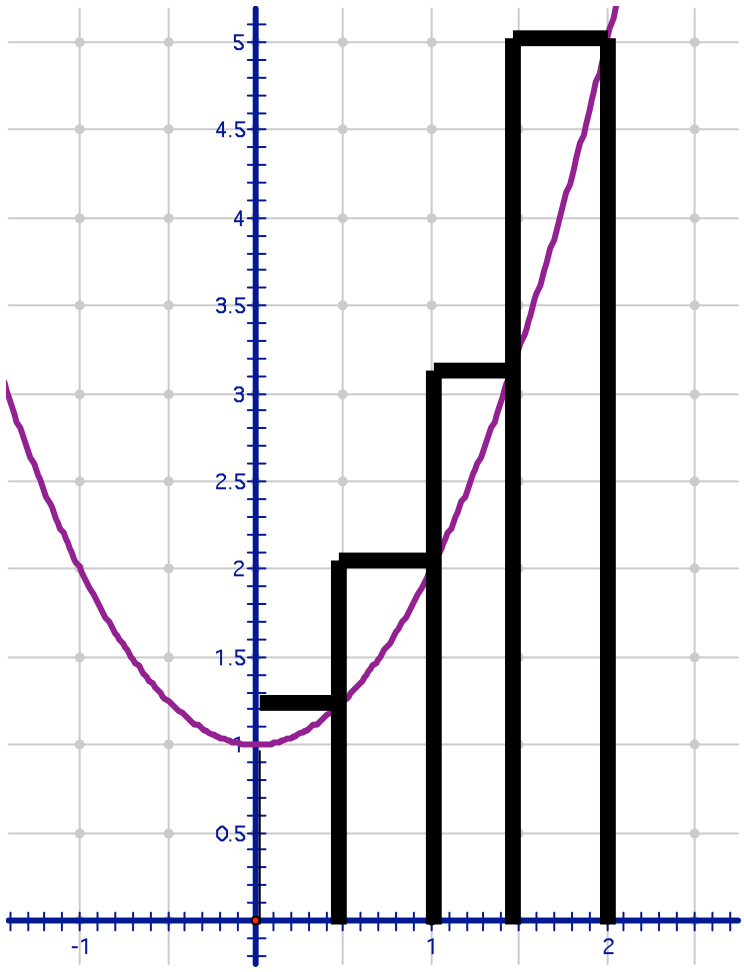
$$L = \frac{1}{2}\left[f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right)\right]$$

$$L = \frac{1}{2}\left(1 + \frac{5}{4} + 2 + \frac{13}{4}\right)$$

$$L = \frac{15}{4}$$

For an increasing function, the lower sum will be calculated using the left side of the interval

Upper Sum



$$U = \frac{1}{2}\left[f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2)\right]$$

$$U = \frac{1}{2}\left(\frac{5}{4} + 2 + \frac{13}{4} + 5\right)$$

$$U = \frac{23}{4}$$

For an increasing function, the upper sum will be calculated using the right side of the interval.