Second Fundamental Theorem of Calculus

We are going to look at the integral where the variable is the upper limit. To avoid confusion, we will use t as the variable of integration.

$$\int_a^x f(t)\,dt$$

Let's consider the following integral:

$$\int_2^x t^2\,dt = \frac{1}{3}t^3\Big]_2^x = \frac{1}{3}x^3 - \frac{8}{3}$$

Notice this is a function of x.

So, $\int_a^x f(t)\,dt$ represents a function of x.

Since integral is an antiderivative, we can use $F(x)$.

$$F(x) = \int_a^x f(t)\,dt$$

From our previous example:

$$F(x) = \int_2^x t^2\,dt$$

$$F(x) = \frac{1}{3}x^3 - \frac{8}{3}$$

Now find F'.

$$F'(x) = x^2$$

This is just $f(t)$ evaluated at the upper limit.

This leads us to the **Second Fundamental Theorem of Calculus**:

If $F(x) = \int_a^x f(t)\,dt$, then $F'(x) = f(x)$.

or

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

This means every continuous function has an antiderivative.

Example: For the following function, find F' in two ways.

$$F(x) = \int_0^x \cos t\,dt$$

By the Second FTC:

$$F'(x) = \cos x$$

Long way:

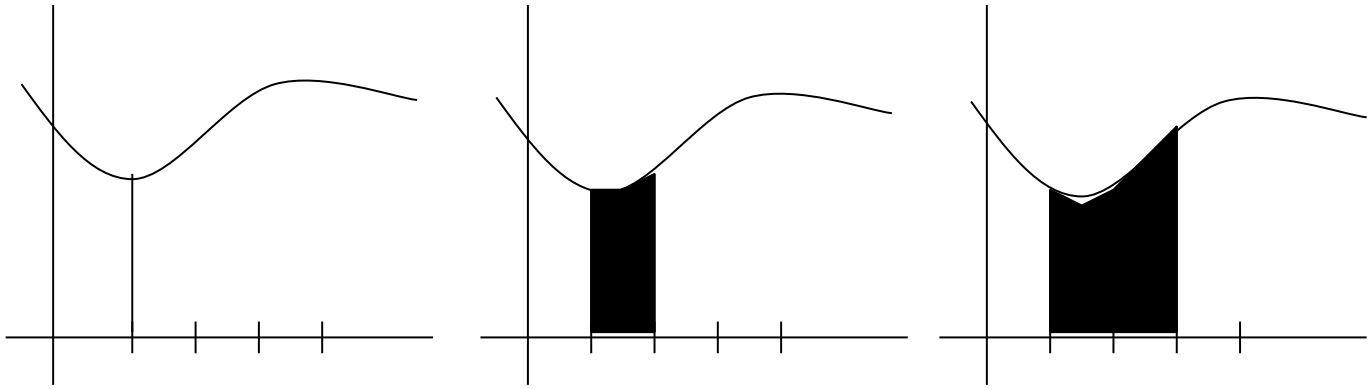
$$F(x) = \int_0^x \cos t\,dt = \sin t\Big]_0^x = \sin x - \sin 0 = \sin x$$

$$F(x) = \sin x$$

$$F'(x) = \cos x$$

What happens as x varies?

$$F(x) = \int_2^x f(t)\,dt$$



$F(2)$ $F(4)$ $F(6)$

As x increases, the area under the curve increases. We are accumulating area. The integral is an accumulation function.

For each of the following, find F'.

Example: $F(x) = \int_1^x \frac{\sqrt{3t^6 + 7}}{t}\, dt$

$$F'(x) = \frac{\sqrt{3x^6 + 7}}{x}$$

Example: $F(x) = \int_1^{x^3} t^4 dt$

Derivative of x^3

$$F'(x) = (x^3)^4 (3x^2) = 3x^{14}$$

This second example involves the chain rule. If the upper limit has a derivative that is not equal to 1, then we have to use the chain rule.

Example: $F(x) = \int_1^{\cos x} t^4 \, dt$

Derivative of $\cos x$

$$F'(x) = (\cos x)^4(-\sin x) = -\sin x \cos^4 x$$

Again, we have to take the derivative of $\cos x$, since that is the upper limit.