Functions and Their Inverses

Consider the following functions:

$$f(x) = 2x - 4 \qquad\qquad g(x) = \frac{1}{2}x + 2$$

$$f(0) = -4 \qquad\qquad g(-4) = 0$$

$$f(5) = 6 \qquad\qquad g(6) = 5$$

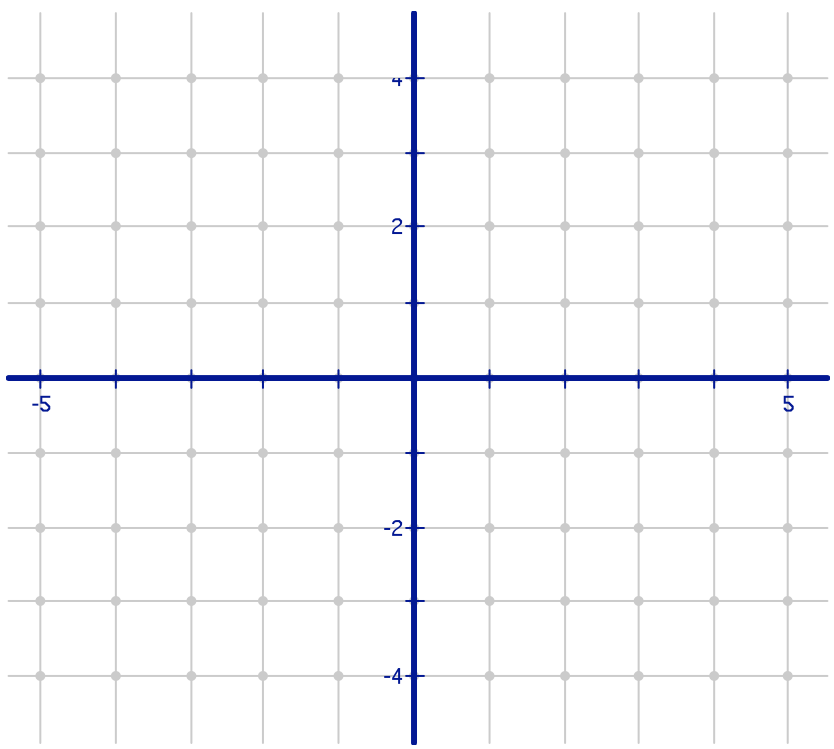
These two functions are inverses:

$$f(x) = g^{-1}(x) \qquad \text{and} \qquad g(x) = f^{-1}(x)$$

Formally, if two functions are inverses, then

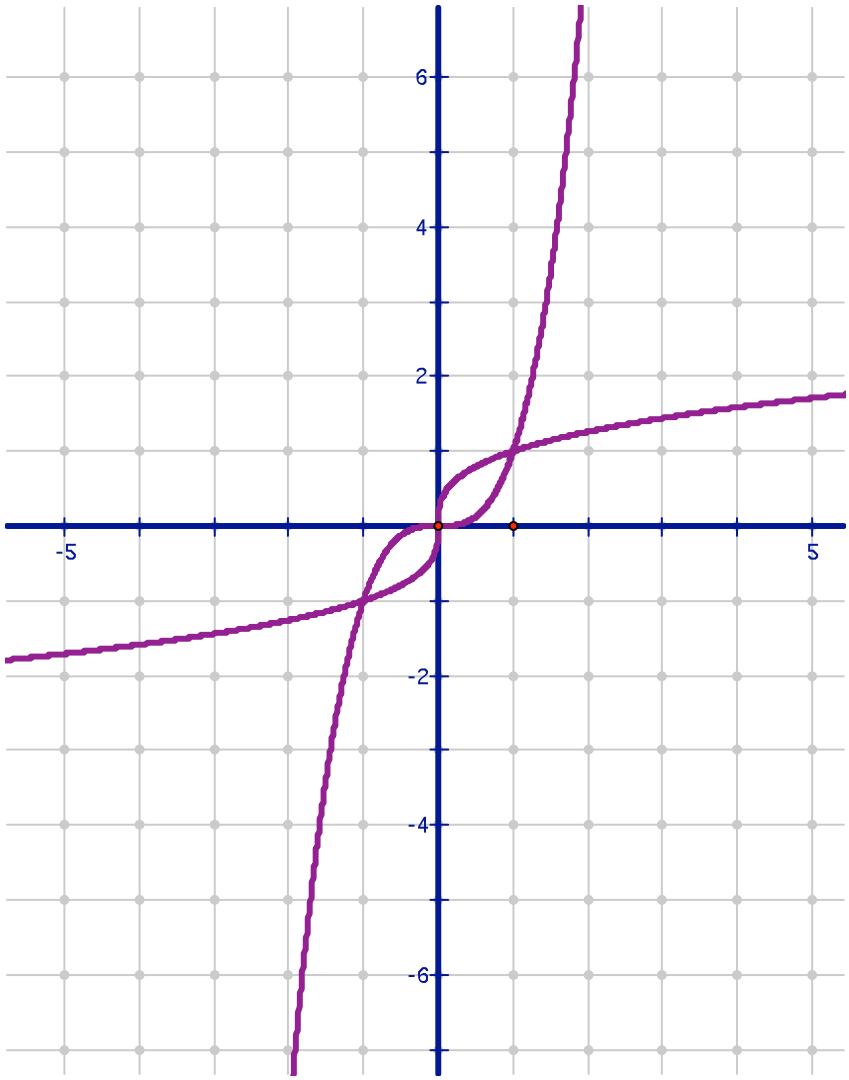
$$f(g(x)) = x \qquad \text{and} \qquad g(f(x)) = x.$$

What about graphically?



Let's try this one. Graph the following function and its inverse.

$$y = x^3$$



It looks like they are symmetric with respect to the line $y = x$.
What about their slopes? How do they compare?

Let's consider the slope of the function at $(2, 8)$ and the slope of the inverse at $(8, 2)$.

$$f(x) = x^3 \qquad f^{-1}(x) = x^{1/3}$$

$$f'(x) = 3x^2 \qquad (f^{-1})'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$$

$$f'(2) = 12 \qquad (f^{-1})'(2) = \frac{1}{12}$$

The slopes of corresponding points are reciprocals.

Not all functions have inverses. For a function to have an inverse, it must pass the **horizontal line test**.

Using calculus, we say that a function has an inverse if it is monotonic. **Monotonic** means it is always increasing or always decreasing. We can use the derivative!

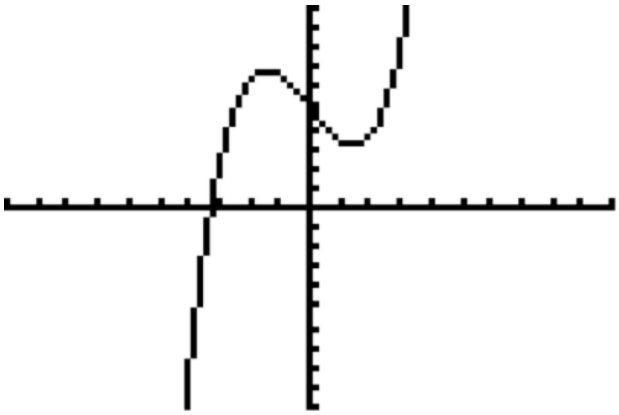
We use the first derivative to decide if the function is strictly monotonic on its entire domain and therefore has an inverse function. Let's do some examples.

Example 1: $y = \frac{x^3}{3} - 2x + 5$

$$y' = x^2 - 2$$

$$x^2 - 2 > 0$$

$$x > \sqrt{2} \text{ or } x < -\sqrt{2}$$



Graph fails horizontal line test.

The derivative is not always positive or negative, so this function is not monotonic and does not have an inverse. We can show this both analytically, using the derivative, as well as graphically, using the horizontal line test.

Example 2: $f(x) = 5 - x - x^3$

$$f'(x) = -1 - 3x^2$$

Since $-1 - 3x^2$ is always less than 0, the function is monotonic and has an inverse.

Example 3: $f(x) = \ln(x - 3)$

$$f'(x) = \frac{1}{x - 3}$$

Notice on this function that the domain is $x > 3$. This means the derivative is always positive, so the function is monotonic and has an inverse. It wouldn't work without the domain restriction.