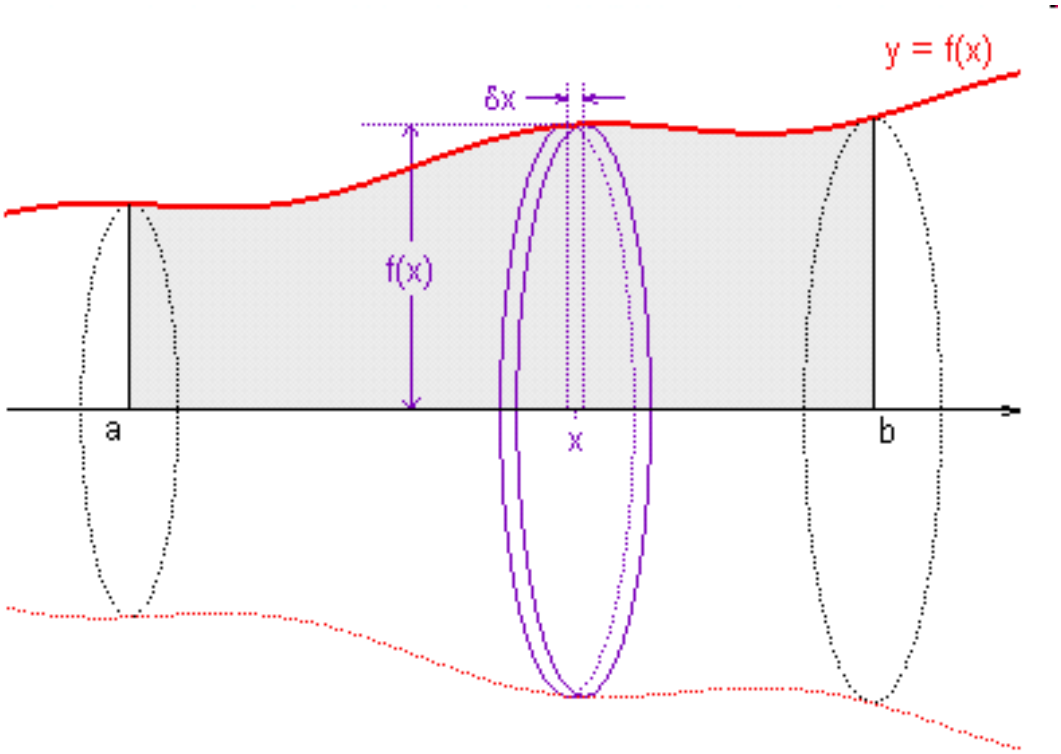
Volumes of Revolution: The Disk Method



If you consider the rectangles we have accumulated in order to get the area, we can get volume by rotating the rectangles about an axis of revolution as in the image above. Notice the shape created by rotating the rectangle is a disk. If we consider the area of each disk and then add them together, we get the volume. Since the disk is circular, its area will be

$$A = \pi r^2 .$$

The radius of the circle is the distance from the axis of revolution to the function. We will start out by revolving them around the x-axis, and move on to revolving them around different lines. Each time, we have to consider the radius.

Formulas Used in the Disk Method

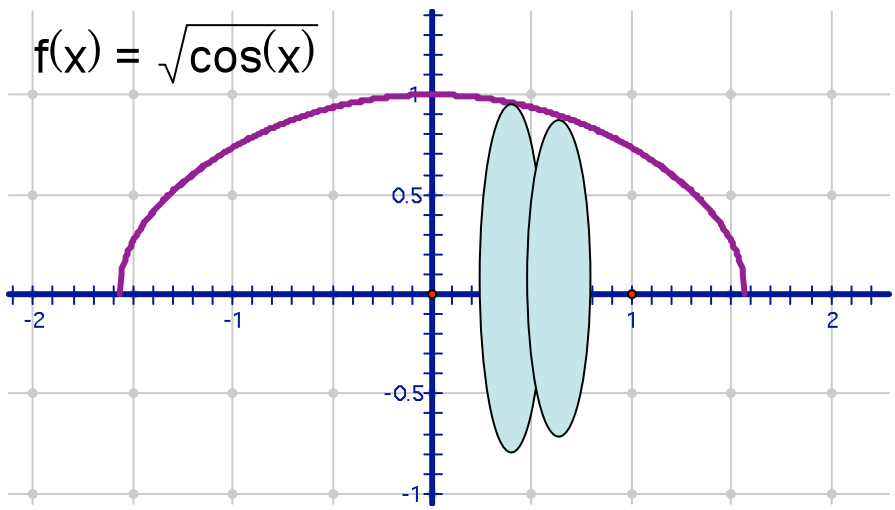
Horizontal axis of revolution:

$$V = \pi \int_a^b [R(x)]^2 \, dx$$

Vertical axis of revolution:

$$V = \pi \int_c^d [R(y)]^2 \, dy$$

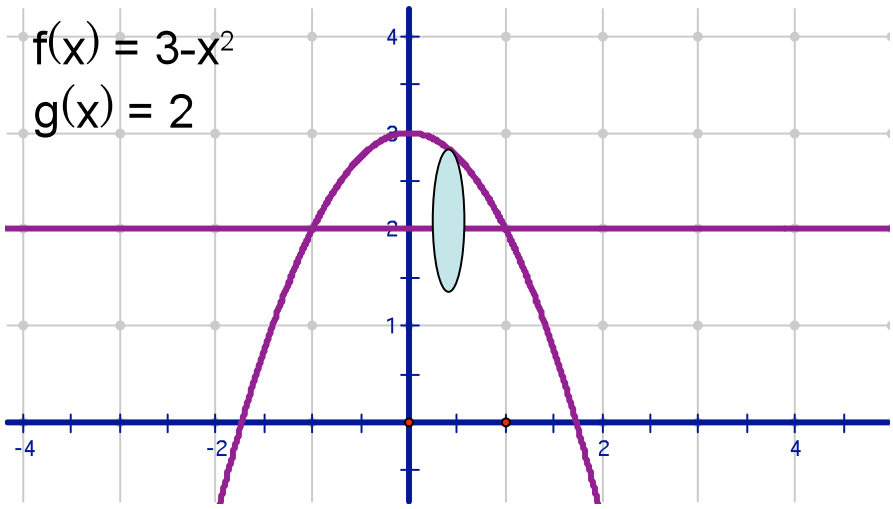
Example 1: Find the volume of the solid formed by $f(x) = \cos x$ and the x-axis on $[-\pi/2, \pi/2]$.



The radius of the circle is $\cos x$. So the integral is the following:

$$
\begin{aligned}
V &= \pi \int_{-\pi/2}^{\pi/2} (\sqrt{\cos x})^2 \, dx \\
&= \pi \int_{-\pi/2}^{\pi/2} \cos x \, dx \\
&= \pi \Big[\sin x \Big]_{-\pi/2}^{\pi/2} \\
&= \pi [1 - (-1)] \\
&= 2\pi
\end{aligned}
$$

Example 2: Find the volume of the solid formed by revolving the region bounded by $f(x) = 3 - x^2$ and $g(x) = 2$ about the line $y = 2$.



$$\boxed{\begin{aligned} 3 - x^2 &= 2 \\ -x^2 &= -1 \\ x &= \pm 1 \end{aligned}}$$

We will integrate from -1 to 1. The radius of the circle will be $f(x) - g(x)$.

$$\begin{aligned}
V &= \pi \int_{-1}^{1} [(3 - x^2) - 2]^2 \, dx \\
&= \pi \int_{-1}^{1} (1 - x^2)^2 \, dx \\
&= \pi \int_{-1}^{1} (1 - 2x^2 + x^4) \, dx \\
&= \pi \left[x - \frac{2}{3} x^3 + \frac{1}{5} x^5 \right]_{-1}^{1} \\
&= \pi \left[\left(1 - \frac{2}{3} + \frac{1}{5} \right) - \left(-1 + \frac{2}{3} - \frac{1}{5} \right) \right] \\
&= \frac{16}{15} \pi
\end{aligned}$$

Here is a Web site that has a flash applet illustrating volumes of revolution if you would like to take a look.

http://mathforum.org/mathtools/cell/c,15.12.5,ALL,ALL/