Growth and Decay

The statement "y is proportional to x" means that $\frac{y}{x}$ is constant, so

$$\frac{y}{x} = k \qquad \text{or} \qquad y = kx.$$

If the rate of change of y is directly proportional to y (the present amount) then

$$\frac{dy}{dt} = ky.$$

Differential equations describe a lot of stuff, including growth and decay problems.

Let's solve the following equation for y.

$$\frac{dy}{dt} = ky$$

$$\frac{1}{y}dy = kdt$$

$$\int \frac{1}{y}dy = \int kdt$$

$$\ln|y| = kt + C$$

$$|y| = e^{kt+C}$$

$$|y| = e^{kt} \cdot e^{C}$$

$$|y| = Ce^{kt}$$

Does this look familiar?

At $t = 0$, $y = C$ (the initial amount). It can also be written in the following ways:

$$y = y_0 e^{kt}$$

$$A = Pe^{kt}$$

$$N = N_0 e^{kt}$$

Example 1: A fruit fly population grows exponentially. There are 100 after the second day and 300 after the fourth day. Find k and use it to find the original number of flies.

There are two ways to find k:

First way:

$$100 = Ce^{2k}$$

$$300 = Ce^{4k}$$

$$\frac{300}{100} = \frac{Ce^{4k}}{Ce^{2k}}$$

$$3 = e^{2k}$$

$$\ln 3 = 2k$$

$$k = \frac{\ln 3}{2} \approx 0.5493061443$$

Second way:

$$300 = 100e^{2k}$$

$$3 = e^{2k}$$

$$\ln 3 = 2k$$

$$k = \frac{\ln 3}{2}$$

We get the same thing either way.

Now that we know k, we can find C, the original number of flies.

$$100 = Ce^{\frac{\ln 3}{2}\cdot 2}$$

$$100 = Ce^{\ln 3}$$

$$100 = 3C$$

$$C = \frac{100}{3}$$

$$\boxed{y = \frac{100}{3}e^{\frac{\ln 3}{2}t}}$$

Standard Equation (even though the question doesn't ask for it)

Example 2: Ten grams of plutonium isotope Pu-239 was released in Chernobyl. If the half-life of Pu-239 is 24,360 years, how long will it take for the original 10 grams to decay to 1 gram?

First, we need to find k. We use 5, because it is half of 10 and we are given information about the half-life.

$$y = Ce^{kt}$$

$$5 = 10e^{24,360k}$$

$$\frac{1}{2} = e^{24,360k}$$

$$\ln\frac{1}{2} = 24,360k$$

$$k = \frac{\ln\frac{1}{2}}{24,360}$$

$$y = 10e^{\frac{\ln\frac{1}{2}}{24,360}\cdot t}$$

To determine how long it will take to make $y = 1$, we solve for t.

$$y = 10e^{\frac{\ln\frac{1}{2}}{24,360}\cdot t}$$

$$1 = 10e^{\frac{\ln\frac{1}{2}}{24,360}\cdot t}$$

$$\frac{1}{10} = e^{\frac{\ln\frac{1}{2}}{24,360}\cdot t}$$

$$\ln\frac{1}{10} = \frac{\ln\frac{1}{2}}{24,360}\cdot t$$

$$t = \left(\ln\frac{1}{10}\right)\cdot\left(\frac{24,360}{\ln\frac{1}{2}}\right)$$

$$t \approx 80,922 \text{ years}$$