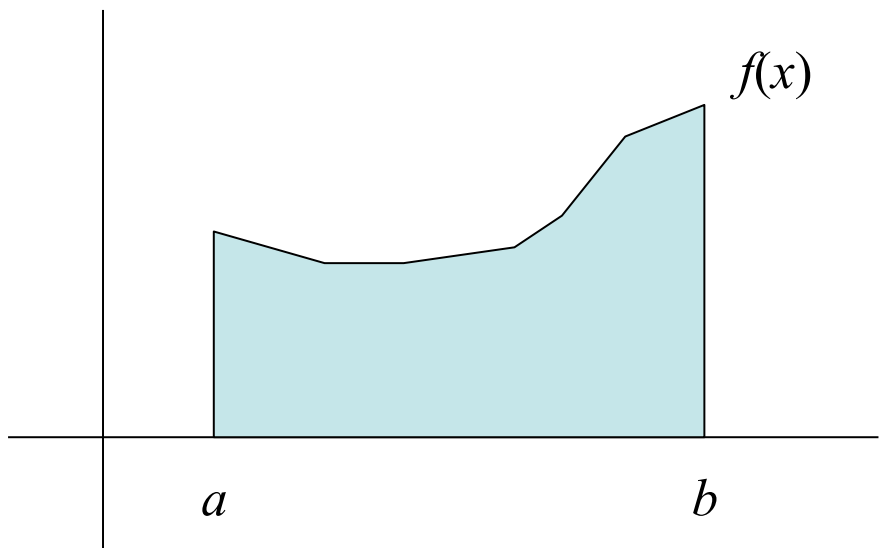
The Area Between Two Curves

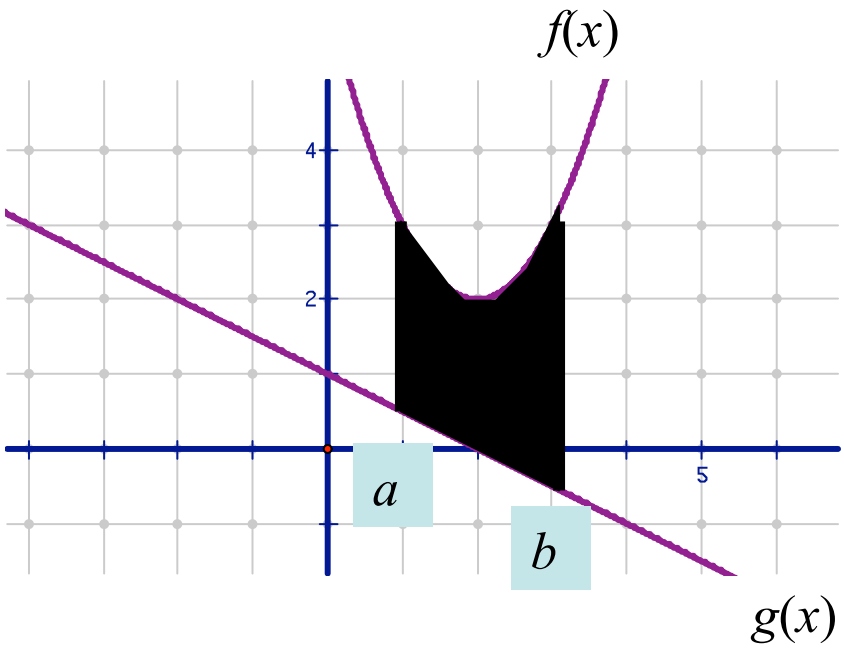
Up to this point, we have found the area between a curve and the x-axis, as illustrated below:



The width of the rectangles is dx and the height of each rectangle is $f(x)$.

$$A = \int_a^b f(x)\,dx$$

Now, we are going to find the area between two curves.



The width of each rectangle is still dx, but now, the height of the rectangle is $f(x) - g(x)$. This means our new area is

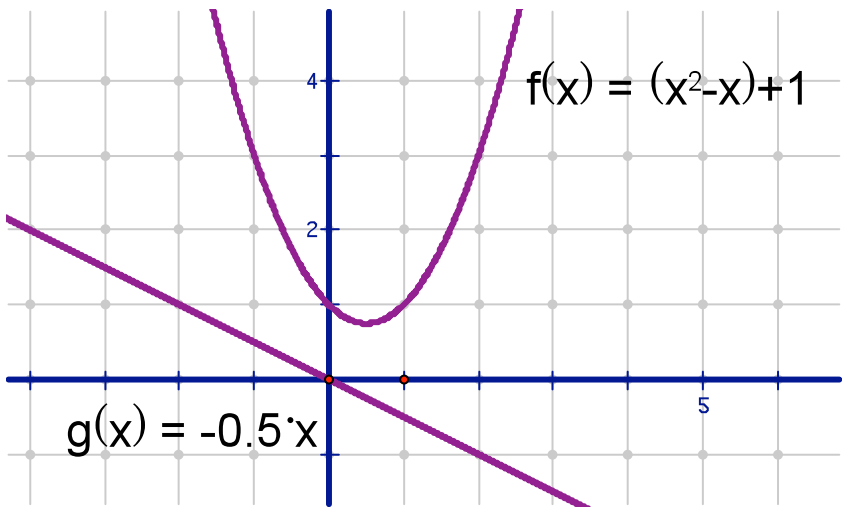
$$A = \int_a^b [f(x) - g(x)]dx.$$

The function $f(x)$ goes first because on our interval, $f(x)$ is greater than $g(x)$.

The formal theorem is found on textbook page 413.

Example 1: Find the area between the following functions on $[-1, 2]$.

$$f(x) = x^2 - x + 1 \qquad \text{and} \qquad g(x) = -\frac{1}{2}x$$



From the picture, we can see that on the interval, $f(x)$ is greater than $g(x)$, so we will calculate the following:

$$\int_{-1}^{2}[f(x) - g(x)]dx = \int_{-1}^{2}\left[\left(x^2 - x + 1\right) - \left(-\frac{1}{2}x\right)\right]dx$$

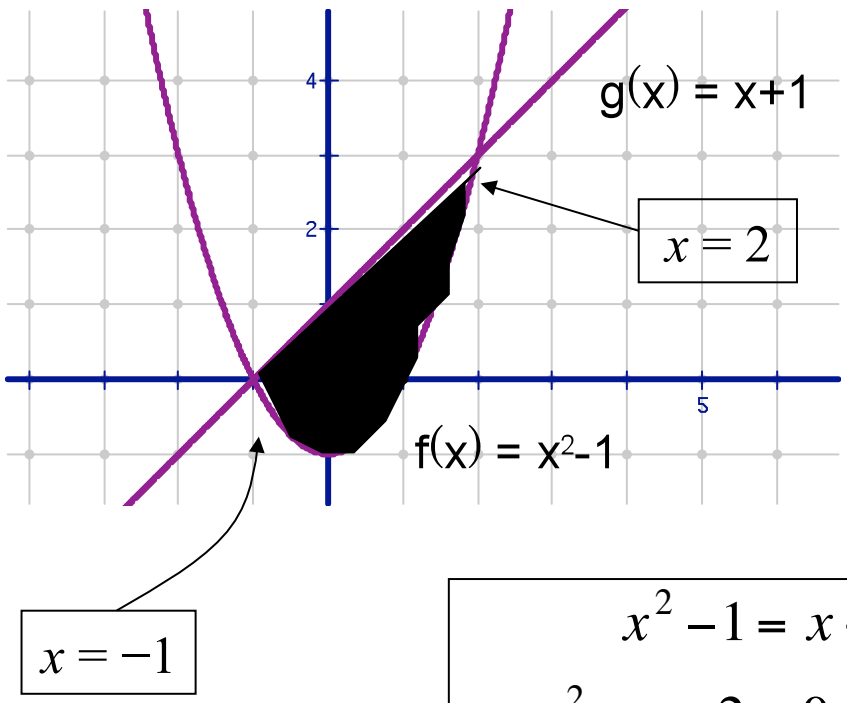
$$= \int_{-1}^{2}\left(x^2 - \frac{1}{2}x + 1\right)dx$$

$$= \left[\frac{1}{3}x^3 - \frac{1}{4}x^2 + x\right]_{-1}^{2}$$

$$= \left(\frac{8}{3} - 1 + 2\right) - \left(-\frac{1}{3} - \frac{1}{4} - 1\right)$$

$$= \frac{21}{4}$$

What if our curves intersect? In this case, we have to calculate the x-values of the points of intersection and they will be the bounds for integrating. Consider the following graph:

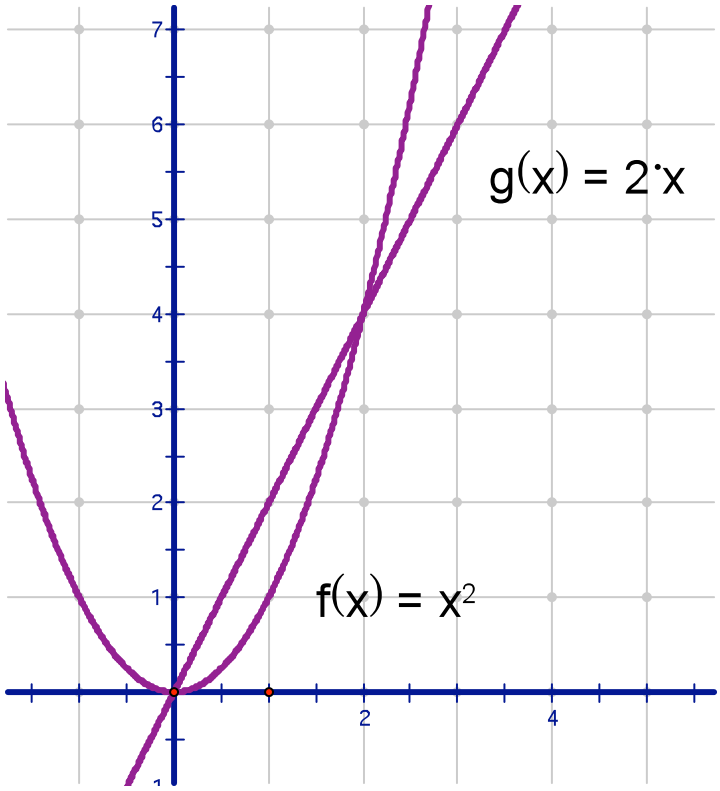


$$\begin{aligned} x^2 - 1 &= x + 1 \\ x^2 - x - 2 &= 0 \\ (x-2)(x+1) &= 0 \\ x = 2 \quad x &= -1 \end{aligned}$$

So we will integrate on $[-1, 2]$. Since $g(x)$ is greater than $f(x)$, the integral will look like this:

$$\int_{-1}^{2} [(x+1) - (x^2 - 1)]dx.$$

Example 2: Find the area of the region bounded by the graphs of $f(x) = x^2$ and $g(x) = 2x$.



$$\boxed{\begin{aligned} x^2 &= 2x \\ x^2 - 2x &= 0 \\ x(x-2) &= 0 \\ x = 0 \quad x &= 2 \end{aligned}}$$

$$\int_0^2 (2x - x^2)\,dx = \left[x^2 - \frac{1}{3}x^3\right]_0^2$$

$$= \left(4 - \frac{8}{3}\right) - (0 - 0)$$

$$= \frac{4}{3}$$