Integrating Inverse Trig Functions

Recall the derivative rules:

$$\frac{d}{dx}[\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$$

$$\frac{d}{dx}[\arctan u] = \frac{u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arccot} u] = \frac{-u'}{1+u^2}$$

$$\frac{d}{dx}[\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$$

$$\frac{d}{dx}[\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$$

Notice that they fall into 3 pairs of opposites.

When listing the antiderivative that corresponds to each of the inverse trig functions, we only need one from each pair. For example:

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

as opposed to $\qquad -\arccos x + C$

Here are the rules:

1. $$\int \frac{du}{\sqrt{a^2-u^2}} = \arcsin\frac{u}{a} + C$$

2. $$\int \frac{du}{a^2+u^2} = \frac{1}{a}\arctan\frac{u}{a} + C$$

3. $$\int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\operatorname{arcsec}\frac{|u|}{a} + C$$

Integrate. The answers are provided on the following pages.

1. $\int \frac{3}{\sqrt{1-4x^2}}\,dx$

2. $\int \frac{4}{1+9x^2}\,dx$

3. $\int_0^1 \frac{dx}{\sqrt{4-x^2}}$

4. $\int_1^2 \frac{1}{3+(x-2)^2}\,dx$

Answers:

1. $$\int \frac{3}{\sqrt{1-4x^2}}\,dx = 3\int \frac{1}{\sqrt{1-(2x)^2}}\,dx$$

$$= \frac{3}{2}\int \frac{1}{\sqrt{1-u^2}}\,du$$

$$= \frac{3}{2}\arcsin\frac{u}{1} + C$$

$$= \frac{3}{2}\arcsin(2x) + C$$

Let $u = 2x$
$du = 2\,dx$

2. $$\int \frac{4}{1+9x^2}\,dx = 4\int \frac{1}{1+(3x)^2}\,dx$$

$$= \frac{4}{3}\int \frac{1}{1+u^2}\,du$$

$$= \frac{4}{3}\arctan\frac{u}{1} + C$$

$$= \frac{4}{3}\arctan(3x) + C$$

Let $u = 3x$
$du = 3\,dx$

3. $$\int_0^1 \frac{dx}{\sqrt{4-x^2}} = \arcsin\frac{x}{2}\Big]_0^1$$

$$= \arcsin\frac{1}{2} - \arcsin 0$$

$$= \frac{\pi}{6} - 0$$

$$= \frac{\pi}{6}$$

4. $$\int_1^2 \frac{1}{3+(x-2)^2}\,dx = \frac{1}{\sqrt{3}}\arctan\frac{x-2}{\sqrt{3}}\Big]_1^2$$

$$= \frac{1}{\sqrt{3}}\left(\arctan\frac{2-2}{\sqrt{3}} - \arctan\frac{1-2}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\left(\arctan 0 - \arctan\frac{-1}{\sqrt{3}}\right)$$

$$= \frac{1}{\sqrt{3}}\left(0 + \frac{\pi}{6}\right)$$

$$= \frac{\pi}{6\sqrt{3}}$$